OPTIMAL THRUST ALLOCATION FOR TBM INTERCEPTOR MIDCOURSE GUIDANCE

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Abstract

Interceptors for tactical ballistic missile defense typically are conceived to have midcourse phases that make corrections to the original interceptor free-flight path based on updated threat state estimates from the filter associated with a remote sensor. Some concepts call for one midcourse correction, while others call for more frequent corrections. The goal of this study is to find the optimal frequency of midcourse corrections from the point of view of minimizing the terminal error, as well as to determine, for a given design, the optimal allocation of thrust resources. It is found that the more frequently the corrections are made, the less the errors are that are handed over to the terminal phase. Furthermore, even when less fuel is available than that required to take out all known errors, the optimal strategy is to make corrections as soon as the amount of correction required just equals the amount of divert available for each burn, until midcourse divert fuel is exhausted. If continuous thrust is available with a throttlable rocket, then the optimal strategy is almost always to continuously make corrections to keep the known errors at zero, until fuel is exhausted or terminal homing commences.

Introduction

In order for the performance of an exoatmospheric tactical ballistic missile (TBM) interceptor to be maximized, the amount of fuel available for midcourse divert must be balanced between having sufficient fuel to take out all errors and minimizing kill vehicle weight for increased interceptor speed and reach. Furthermore, for a given amount of divert capability, it is desirable to know what the optimal strategy is for scheduling midcourse corrections. Whether there should be one large midcourse correction or numerous smaller corrections needs to be known. Also, for a given number of potential burns, the optimal strategy as to when to apply these corrections needs to be determined.

This paper addresses these issues using optimal control theory and parameter optimization techniques. First, it is assumed for this study that a sensor that is remote from the interceptor is observing the threat at regular intervals, and that the midcourse corrections of the interceptor are based on the remote sensor's threat state estimate. In this mode, the sensor's filter will typically reach a steady state velocity error. This is the basis for a one dimensional motion optimization problem that captures the essence of the full problem. The goal of the problem is to find the optimal thrust strategy that minimizes the terminal error (the error that is handed over for the onboard sensor/filter/divert system to take out in terminal homing, by some guidance method such as proportional navigation[1]).

It is found that in almost all cases the optimal strategy is to make either continuous (if possible), or else as frequent as possible, small corrections all the way to terminal homing. The difference between numerous corrections and one or a small number of corrections can be significant. Not only does this make successful intercept more probable for a given amount of divert, but it also requires significantly less divert capability for a given design goal.

Problem Formulation

It is assumed that the remote sensor's filter has reached steady state during the time period that the midcourse corrections are to be applied. The midcourse corrections are based on propagating the threat state estimate forward from the current time to the time of intercept. It is also assumed that the intercept time does not change much during the correction process, so for this study the intercept time is considered fixed. Given these things, the error in the propagation of the threat state estimate is approximated well by a linear function of time. That is, if σ_v represents some reasonable upper bound on the filter velocity error (such as a 3-

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standard deviation value under a Gaussian probability density function assumption), then a very good estimate of an upper bound for the position error of the propagated threat at intercept time is σ_v times the time-to-go until intercept (with the initial position error being roughly ignorable).

Let t_f be the time from the start of the midcourse phase until the beginning of terminal homing, and Δt_h the duration of terminal homing, so that

$$\tau = t_f + \Delta t_h$$

is the intercept time relative to the beginning of midcourse. If $x(t)$ and $v(t)$ represent the amount of corrective position and velocity made by time t, then the intercept error handed over to the terminal phase is approximated by

$$\sigma_v \tau \ - \ (x(t_f) \ + \ v(t_f) \cdot \Delta t_h) \ .$$

The goal of the optimization process is to minimize this error.

In order for the optimization process to make sense, however, a constraint must be added to the optimization problem to account for the fact that no greater correction can be made at any given moment than the amount of increased knowledge of the threat state, gained over the time span t. Since $\sigma_v (\tau - t)$ is the propagated error at time t, and since $\sigma_v \cdot \tau$ is the initial propagated error, the difference between these, $\sigma_v \cdot t$ can be thought of as the amount of required correction that is known at time t. Therefore, the constraint is that the position of the interceptor at intercept time, if the interceptor coasts for the remainder of the time to go, cannot be greater than the currently known correction amount, $\sigma_v \cdot t$. That is,

$$x(t) + v(t)(\tau - t) - \sigma_v t \leq 0 \ . \qquad (1)$$

Two different formulations will be used for the dynamics. First, a throttlable rocket will be assumed, so that

$$\dot{v} = \frac{\beta \, u_e}{m} \ ,$$

where $\beta \in [0, \beta_{\mathrm{m}}]$ is the mass flow rate, u_e is the equivalent exhaust velocity of the rocket, and m is the current vehicle mass. This leads to an optimal control problem, with β being the control variable.

Second, a series of instantaneous impulse rocket burns will be assumed, to approximate burns of short duration separated by coasting

times. In this case, the optimization process is to choose the optimal instantaneous velocity corrections and corresponding times of the corrections in order to minimize the same cost function, subject to the constraint that the sum of the velocity corrections cannot exceed the total allotted, and subject to Ineq. (1) written in terms of the sum of the velocity corrections and the time increments.

Continuous Control

Optimal Control Formulation

The differential equations of the state are:

$$\dot{x} = v \ , \qquad (2)$$

$$\dot{v} = \frac{\beta \cdot u_e}{m} \ , \qquad (3)$$

and

$$\dot{m} = - \beta \ , \qquad (4)$$

with initial conditions

$$x(0) = 0 \ , \qquad (5)$$

$$v(0) = 0 \ , \qquad (6)$$

and

$$m(0) = m_0 \ . \qquad (7)$$

At the given final time, t_f, $x(t_f)$ and $v(t_f)$ are unspecified (that is, they are free to take on any value that minimizes the cost function), but the vehicle final mass

$$m(t_f) \geq m_{\mathrm{min}} \ . \qquad (8)$$

The way this inequality constraint is handled in practice is to assume that $m(t_f)$ is also unspecified, and if the inequality is violated in the solution, then the solution is attempted again with the equality constraint $m(t_f) = m_{\mathrm{min}}$.

Then the problem is

$$\min_{\beta} \left\{ \sigma_v \tau \ - \ (x(t_f) \ + \ v(t_f) \cdot \Delta t_h) \right\} \qquad (9)$$

subject to the control variable constraint

$$\beta \in [0, \beta_{\mathrm{m}}]$$

and the state constraint

$$Q(t) \leq 0 \ , \qquad (10)$$

where

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$$Q(t) = x(t) + v(t) \cdot (\tau - t) - \sigma_v \cdot t \ . \quad (11)$$

In order to solve this optimal control problem, first the Hamiltonian is set up in the usual manner[2],

$$H = \lambda_x \cdot v + \lambda_v \cdot \frac{\beta u_e}{m} - \lambda_m \beta \ , \quad (12)$$

where the costate variables λ_x, λ_v, and λ_m satisfy the differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 \ , \quad (13)$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_x \ , \quad (14)$$

and

$$\dot{\lambda}_m = -\frac{\partial H}{\partial m} = \frac{\beta u_e}{m^2} \cdot \lambda_v \ . \quad (15)$$

The endpoint conditions on these costate variables are determined by the transversality conditions.

Since all three state variables are specified at the beginning time, transversality states that the costate variables at this time are free (that is, they are free to take on whatever value is required to solve the resultant boundary value problem). At the final time, t_f, each λ_i (for $i \in \{x, v, m\}$) must satisfy

$$\lambda_i(t_f) = \frac{\partial \phi}{\partial i} + \frac{\partial \psi}{\partial i} \cdot \nu \ ,$$

where

$$\phi = \sigma_v \tau - \left(x(t_f) + v(t_f) \cdot \Delta t_h \right)$$

is the Mayer cost term, ψ is the terminal condition vector, and ν is a free parameter to be determined to satisfy the resulting conditions. Since $x(t_f)$ and $v(t_f)$ are free parameters, ψ will not contain any x or v values in it. For the case where the final mass is free,

$$\psi = 0$$

(that is, there are no terminal conditions on the state), and for the case where it is specified,

$$\psi = m - m_f \ .$$

Therefore,

$$\lambda_x(t_f) = -1 \ , \quad (16)$$

$$\lambda_v(t_f) = -\Delta t_h \ , \quad (17)$$

and

$$\lambda_m(t_f) = 0 \quad (18)$$

when the final mass is not specified, or

$$\lambda_m(t_f) = \nu \quad (19)$$

when $m(t_f) = m_{\min}$ (that is, $\lambda_m(t_f)$ is a free parameter).

On points of the trajectory where the state constraint Ineq. (10) is not active, given the state and costate variables on an optimal trajectory, the optimal control value is found through the Minimum Principle, which states that the optimal β satisfies

$$\min_{\beta} H \ .$$

Since Eq. (12) is an affine function in β, the value of the optimal β always lies at one endpoint of the interval $[0, \beta_m]$, unless the slope is zero. Therefore,

$$\beta = \begin{cases} 0 & , S > 0 \\ \beta_m & , S < 0 \end{cases} \ , \quad (20)$$

where

$$S = \frac{u_e \lambda_v}{m} - \lambda_m \ . \quad (21)$$

If $S = 0$ for some finite amount of time, the control is termed singular; this case will be discussed in the last subsection.

Solution Process

The process used to obtain solutions to this problem is a two-step method. First, a differential inclusion direct method is used to obtain numerical estimates of the solutions, with particular emphasis given to finding the appropriate switching structure for each problem[3]. Typically, a multiple shooting method could be used at this point to obtain refined solutions that satisfy the two-point boundary value problem of the six dimensional combined state and costate equations, with state boundary conditions and costate transversality conditions, and incorporating the form of the optimal control in Eq. (20). For this problem, however, the differential equations are simple enough to be integrated analytically for most solution types, so the different switching structures indicated by the numerical solution are analytically pieced

together to obtain appropriate solutions. In some cases the solutions are simple enough, and the logic from the optimal control formulation is clear enough, that the solutions can be concluded without requiring full integration of the two-point boundary value problem.

No Path Constraint Solution

As a first case to solve, assume that there is no path constraint, Ineq. (10), or else that σ_v is sufficiently large compared to β_m such that the constraint can never be reached. The solution to this problem is the simplest of all of the cases (and it is the most intuitively obvious), yet it illustrates many of the methods and ideas needed for the more complicated cases presented later. In this way, some of the more important features of the different types of solutions, as well as the different solution methods, can be presented before the additional complications of state constraints and singular arcs are introduced.

The solution to the differential equation for λ_x , Eq. (13), is a constant, and the transversality condition, Eq. (16), implies that

$$\lambda_x(t) = -1 \qquad (22)$$

as long as there are no jumps in the adjoint variables. Since jumps in the adjoints can only occur when a constraint is encountered, Eq. (22) is the correct solution in this case. Similarly, straightforward integration of Eq. (14) with the application of the terminal boundary condition, Eq. (17), results in

$$\lambda_v(t) = t - \tau . \qquad (23)$$

The solution for λ_m is more complicated since the differential equation Eq. (15) involves $\beta(t)$ and $m(t)$, so the switching structure for $\beta(t)$ must be known. To discern the correct switching structure, consider the time derivative of the switching function, Eq. (21), which is

$$\dot{S} = -\frac{u_e}{m}\lambda_x + \frac{u_e\lambda_v}{m^2}\beta - \frac{\beta u_e}{m^2}\lambda_v .$$

Or,

$$\dot{S} = -\frac{u_e}{m}\lambda_x . \qquad (24)$$

Since $u_e > 0$, $m > 0$, and $\lambda_x < 0$,

$$\dot{S} > 0 . \qquad (25)$$

In other words, $S(t)$ is monotonically increasing. A typical solution of $S(t)$ is shown in Figure 1.

Combining the monotonicity of $S(t)$ with Eq. (20), $\beta(t)$ is either full on all the way (so long as Ineq. (8) is not violated), or else it is full on to full off at the time that $S(t)$ passes through zero.
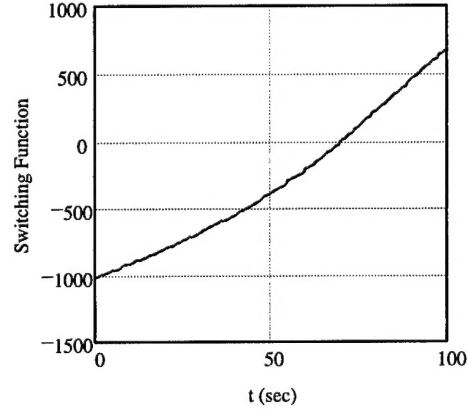

Figure 1. A Typical Switching Function Solution.

For the case when $m(t_f)$ is free, $\lambda_m(t_f) = 0$ from transversality (Eq. (18)). Using this plus the transversality condition on λ_v (Eq. (17)) in Eq. (21) at the final time,

$$S(t_f) = -\frac{u_e\,\Delta t_h}{m(t_f)} ,$$

which implies that

$$S(t_f) < 0 .$$

Combined with the monotonicity of $S(t)$, this means that the optimal solution for the final-mass free case is to thrust at

$$\beta(t) = \beta_m \ \ \forall \ t \in [0, t_f] .$$

For the case when $m(t_f)$ is specified, $\lambda_m(t_f)$ is free; so the terminal condition on the mass is what determines the control. Since the only possible solution is full-on to full-off, it must be that

$$\beta(t) = \beta_m$$

until some switching time, say t_s , at which $m(t_s) = m_{\min}$. Or,

$$m_0 - \beta_m t_s = m_{\min} ,$$

so that

$$t_s = \frac{m_0 - m_{\min}}{\beta_m} .$$

With this knowledge of the switching structure, $\lambda_m(t)$ can be solved for the two cases.

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For all of t for the first case, and for $t \leq t_s$ for the second case, the differential equation for λ_m is

$$\dot{\lambda}_m = \frac{\beta_{\mathrm{m}} u_e (t - \tau)}{(m_0 - \beta_{\mathrm{m}} t)^2} \ ,$$

which can be directly integrated to yield

$$\lambda_m(t) = \frac{u_e}{\beta_{\mathrm{m}}} \left[ln(m_0 - \beta_{\mathrm{m}} t) + \frac{(m_0 - \beta_{\mathrm{m}} \tau)}{(m_0 - \beta_{\mathrm{m}} t)} \right] + C \ .$$

For the final mass free case, using Eq. (18) to determine the constant of integration yields

$$\lambda_m(t) = \frac{u_e}{\beta_{\mathrm{m}}} \left[ln\left(\frac{m_0 - \beta_{\mathrm{m}} t}{m_0 - \beta_{\mathrm{m}} t_f} \right) + \frac{(m_0 - \beta_{\mathrm{m}} \tau)\beta_{\mathrm{m}}(t - t_f)}{(m_0 - \beta_{\mathrm{m}} t)(m_0 - \beta_{\mathrm{m}} t_f)} \right] .$$

For the final mass specified case, $S(t_s) = 0$ determines C. $S(t_s) = 0$ implies that

$$\lambda_m(t_s) = \frac{u_e(t_s - \tau)}{m_{\min}}$$

so that

$$\lambda_m(t) = \frac{u_e}{\beta_{\mathrm{m}}} \left[ln\left(\frac{m_0 - \beta_{\mathrm{m}} t}{m_0 - \beta_{\mathrm{m}} t_s} \right) + \frac{(m_0 - \beta_{\mathrm{m}} \tau)\beta_{\mathrm{m}}(t - t_s)}{(m_0 - \beta_{\mathrm{m}} t)(m_0 - \beta_{\mathrm{m}} t_s)} \right]$$

$$+ \frac{u_e(t_s - \tau)}{m_{\min}}$$

for $t \leq t_s$, and

$$\lambda_m(t) = \frac{u_e(t_s - \tau)}{m_{\min}}$$

for $t > t_s$

The conclusion, then, is that when there is no path constraint (which is only of academic interest), or when the path constraint can never be reached for a given β_{m} limit on the mass flow rate (which signifies that that sensor/filter errors are too large for the given divert capability), the optimal strategy is to thrust at the maximum level until either the start of terminal homing or the fuel available is exhausted. The former case is self evident. The latter case results because the earlier a correction is made, the more the terminal error that can be taken out, because the velocity times time-to-go is greater.

Costate Jumps Upon Hitting the State Constraint

For all cases other than the cases where there is no state path constraint or where the constraint cannot be reached, the possibility of being on a trajectory that hits the constraint must be considered. In general, the costates and the Hamiltonian (and hence the switching function

$S(t)$) will be discontinuous at the time when the trajectory encounters the state constraint[4]. The discontinuity for λ_x at the time the constraint is entered is

$$\lambda_x^- = \lambda_x^+ + \mu \frac{\partial Q}{\partial x} \ ,$$

or

$$\lambda_x^- = \lambda_x^+ + \mu \ . \tag{26}$$

Similarly,

$$\lambda_v^- = \lambda_v^+ + \mu(\tau - t) \tag{27}$$

and

$$\lambda_m^- = \lambda_m^+ \ . \tag{28}$$

Also, the Hamiltonian, Eq. (12), must satisfy

$$H^- = H^+ - \mu \frac{\partial Q}{\partial t} \ . \tag{29}$$

Plugging Eq.s (26), (27) and (28) into Eq. (29), using Eq.s (12) and (11), yields the necessary condition

$$\mu \left[\frac{\beta^-(\tau - t)}{m} u_e - \sigma_v \right] = \left[\frac{\lambda_v^+ u_e}{m} - \lambda_m \right] \Delta\beta \ ,$$

where $\Delta\beta$ is the jump in the control at that time. This equation can be satisfied either by

$$\mu = \frac{\lambda_v^+ u_e - m \lambda_m}{\beta^-(\tau - t) u_e - m \sigma_v} \ \Delta\beta \ , \tag{30}$$

or else by

$$\beta^- = \frac{m \sigma_v}{u_e (\tau - t)} \tag{31}$$

combined with

$$\Delta\beta = 0$$

or

$$S^+ = 0 \ .$$

It is interesting to note that Eq. (31) is the same control level required to keep the trajectory on the constraint, as presented in the next section. This means that $\dot{Q} = 0$ at the time the constraint is encountered if the control is continuous across the jump.

Large Mass Flow Rate Solution

Assume now that the mass flow rate is arbitrarily large, so that $\beta_{\mathrm{m}} \to \infty$. This case is of interest for designing a system, in which case the

maximum resulting β_m will guide the design thrust level. And, in practice, this case is solved first, and if a β_m limit is violated, then the problem is solved again with new switching structures consistent with the limit.

The numerical solutions for large mass flow rate cases all show that following the constraint from the start until either time t_f is reached, or else until mass m_{min} is reached, is the optimal solution.

When the constraint is encountered, $Q(t)$ from Eq. (11) must equal zero. Furthermore, $\dot{Q}(t)$ must also be zero to keep the trajectory on the constraint. Setting $\dot{Q}$ to zero and solving for β yields the control value required to keep the trajectory on the constraint. That is,

$$\dot{Q} = \frac{\beta u_e}{m}(\tau - t) - \sigma_v , \qquad (32)$$

so that,

$$\beta = \frac{\sigma_v m}{u_e(\tau - t)} . \qquad (33)$$

Now, $\dot{m} = -\beta$, so

$$\int \frac{dm}{m} = - \int \frac{\sigma_v \, dt}{u_e(\tau - t)}$$

or,

$$m(t) = C \, (\tau - t)^{\sigma_v/u_e}.$$

For the case when the trajectory starts on the constraint, $m(0) = m_0$, so that

$$m(t) = m_0 \left(\frac{\tau}{\tau - t} \right)^{-\sigma_v/u_e} .$$

To solve for the corresponding $v(t)$, the differential equation, Eq. (3), with the right side of Eq. (33) substituted in, becomes

$$\dot{v} = \frac{\sigma_v}{\tau - t} ,$$

which has the solution (with initial condition Eq. (6))

$$v(t) = \sigma_v \cdot ln\left(\frac{\tau}{\tau - t} \right) . \qquad (34)$$

Integrating this to obtain position, $x(t)$, with the initial condition Eq. (5), and using Eq. (34) to simplify,

$$x(t) = \sigma_v t - v(t) \cdot (\tau - t) . \qquad (35)$$

Consider now the necessary conditions of optimal control for this case. On the path

constraint, the $\dot{Q} = 0$ constraint must be adjoined to the Hamiltonian, with the associated Lagrange multiplier $\alpha(t)$, so that

$$H = \lambda_x \cdot v + \lambda_v \cdot \frac{\beta u_e}{m} - \lambda_m \beta$$

$$+ \alpha \cdot \left(\frac{\beta u_e}{m}(\tau - t) - \sigma_v \right) .$$

Since x and v are not in $\dot{Q}$, the λ_x and λ_v differential equations remain unchanged. For λ_m

$$\dot{\lambda}_m = \frac{\beta u_e}{m^2}[\lambda_v + \alpha(\tau - t)] . \qquad (36)$$

$\alpha(t)$ is computed by taking $\frac{\partial H}{\partial \beta}$, setting it to zero, and solving for α, which yields

$$\alpha = \frac{m\lambda_m - u_e\lambda_v}{u_e(\tau - t)} .$$

Plugging this into Eq. (36) yields

$$\dot{\lambda}_m = \frac{\beta \lambda_m}{m} ,$$

which has the solution

$$\lambda_m = C \, (\tau - t)^{-\frac{\sigma_v}{u_e}} . \qquad (37)$$

Consider now the case where the trajectory starts with thrust off, switching to full thrust on until the constraint is hit. When the final mass is free, transversality demands that $\lambda_m(t_f)$ be zero (Eq. (18)). Plugging this into Eq. (37) yields $C = 0$, so

$$\lambda_m(t) = 0$$

is the solution when the constraint is ridden to the final time.

Next, plugging this into Eq. (21) at the time the constraint is hit, t_{hit},

$$S(t_{hit}^+) = \frac{u_e \lambda_v(t_{hit}^+)}{m(t_{hit})} ,$$

and

$$S(t_{hit}^-) = \frac{u_e \lambda_v(t_{hit}^-)}{m(t_{hit})}$$

$$= \frac{u_e}{m}[\lambda_v(t_{hit}^+) + \mu \, (\tau - t_{hit})]$$

$$= \frac{u_e}{m}(\tau - t_{hit}) \, (\mu - 1) .$$

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For the interceptor to have been thrusting when it hit the constraint,

$$S(t_{hit}^-) < 0 \quad \Rightarrow \quad \mu - 1 < 0 ,$$

or

$$\mu < 1 .$$

On the other hand, considering Eq. (21), it is required that $\dot{S} < 0$ for a bang-off to bang-on solution. Examining Eq. (24), this in turn necessitates $\lambda_x^- > 0$. Also, since transversality requires $\lambda_x(t_f) = -1$, Eq. (26) requires

$$\mu > 1 .$$

Since this conflicts with the previous inequality on μ, this switching structure can never happen. Therefore, when the final mass is free, a bang-off to bang-on to constraint solution does not satisfy the necessary conditions for optimal control. Riding the constraint the whole way is the optimal solution.

If the final mass is constrained, the optimal solution is to start on the constraint, and ride it until the available fuel is expended. Showing that this satisfies the necessary conditions is a straightforward piecing together of the adjoint solutions derived above (and the switching function S) for the constrained arc following by the coasting arc. It is also straightforward to show that if the trajectory coasts first, then thrusts at the maximum until the constraint is hit, then this always burns more fuel, requiring the thrust to be terminated earlier and the terminal error to be higher. Therefore riding the constraint from the start until the fuel is expended is the optimal solution.

Thrust-Limited Solutions

Consider now the case when the $\beta_m \to \infty$ solution exceeds the true β_m value. For example, a typical solution for the optimal thrust level (where the thrust $T = \beta \cdot u_e$) for riding the constraint the whole way is shown in Figure 2. The parameters used here are $\sigma_v = 150 \, m/s$; $t_f = 100 \, s$; $\Delta t_h = 10 \, s$; and $u_e = 2455 \, m/s$. Suppose that the maximum thrust is set to 1000 N, which corresponds to $\beta_m = 0.407323 \, kg/s$. Then the time at which the maximum thrust is reached is $t = 75.036 \, s$. One solution that satisfies the necessary conditions for optimal control is to ride the constraint until the time, t_s, that the value of β required to stay on the constraint reaches β_m, then switch to $\beta(t) = \beta_m$

for $t > t_s$. The resulting cost function value for this case is 2793 m.
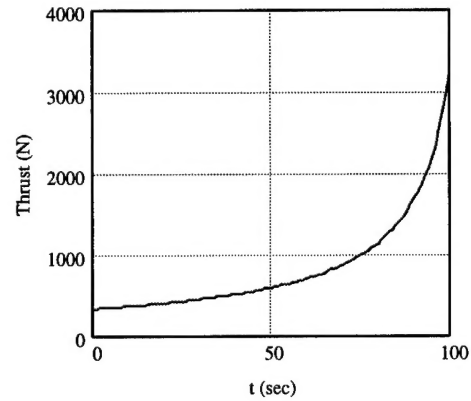

Figure 2. Thrust for the Unlimited Thrust Solution.

For all of the cases examined by the authors, however, a bang-off to bang-on solution always produces slightly lower cost (2770 m for the example presented here). The solution starts with thrust off, switching to $\beta(t) = \beta_m$ at time t_s. The switch time t_s and the jump value μ are iterated upon until Eq. (31) is satisfied at some time t_{hit}, at which time the constraint is just touched as $Q(t)$ reaches a maximum. Fig. 3 gives a plot of $Q(t)$ for the same example mentioned above. The solution parameters are $\mu = 1.0126$, and $t_s = 28.3787$, yielding $t_{hit} = 75.3712$.
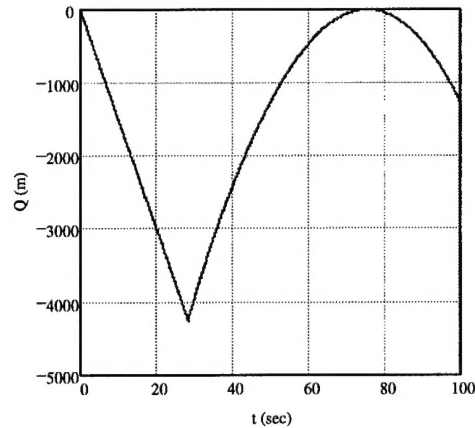

Figure 3. State-Constraint Function for the Bang-Off to Bang-On Solution.

Once the mass constraint, Eq. (8), is encountered in a solution, however, the simple bang-bang solution is no longer an option,

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because the switch time must change in order to burn less fuel, in which case $Q(t)$ can never touch zero, so that the costates can never jump to the values required by terminal transversality. The only other possibility would be to keep the same switching time and terminate the thrust when $m(t) = m_{min}$. This cannot be optimal, though, because $S(t)$ must reach zero at the time the thrust is switched off, but from the mass free solution $S(t_f) < 0$.

For this case it turns out that the optimal solution is to ride the state constraint until β_m is encountered, then ride the β_m constraint until the fuel is exhausted, then set $\beta(t) = 0$ after that. Take the example presented above. The final mass for the mass-free bang-bang solution is $m(t_f) = 220.8\,kg$, with $m_0 = 250\,kg$. If the required final mass is now restricted such that it can be no less than $m_{min} = 225\,kg$, then the solution via riding the constraint has a cost of $3078\,m$. By contrast, the bang-bang cost is $3465\,m$. Also, for comparison purposes, if the β_m restriction is lifted, yet the $m_{min} = 225\,kg$ restriction is maintained, then the optimal solution is to ride the constraint until fuel is exhausted, resulting in a cost of $2942\,m$.

Singular Control

In order for an optimal control study to be complete, the possibility of singular control must be investigated to see if the global optimum includes any singular arcs[5]. For singular control, $S(t)$ must be identically zero for some finite amount of time, so $S(t)$ and all of its derivatives must equal zero on singular arcs. Typically, derivatives of $S(t)$ are set to zero to determine conditions necessary to have singular control; at some derivative the control appears (always at an even derivative), and the resulting equation is used to solve for the value of the control. To this end, consider

$$\dot{S} = -\frac{u_e}{m}\lambda_x \,.$$

Since $u_e > 0$ and $m > 0$, it is required that

$$\lambda_x = 0$$

on singular arcs. Now the second derivative is

$$\ddot{S} = -\frac{u_e}{m^2}\beta\lambda_x \,.$$

Here the control has appeared, but $\lambda_x = 0$, so that any control will satisfy the necessary

conditions. This solution trivially satisfies the generalized Legendre-Clebsch condition[6]

$$(-1)^k\frac{\partial}{\partial\beta}\left(\frac{d^{2k}S}{dt^{2k}}\right) \geq 0 \,, \quad k = 1, 2, \ldots$$

with strict equality since λ_x is a factor in all the derivatives of $S(t)$.

λ_x, however, must be equal to -1 at the terminal time, t_f, so the constraint must be encountered in order for it to jump from 0 to -1. This means that $\mu = 1$ in Eq.s $(26) - (29)$. Therefore, from Eq. (29) with $\mu = 1$,

$$\lambda_v(t) = 0 \,, \quad t < t_{hit} \,,$$

where t_{hit} is the time the constraint is encountered. This in turn implies that (see Eq. (21))

$$S(t) = -\lambda_m(t_{hit}) \,, \quad t < t_{hit} \,,$$

which in general is not equal to zero except for the case where $t_{hit} = t_f$ and final mass is free.

The implication of this is that if the constraint can be hit at the final time by following any control history, it is an optimal control. This is obviously true, since, as mentioned previously, hitting the constraint at the final time, if it is possible to achieve this, is the global optimum. So, for example, if β_m is sufficiently large and if there is sufficient fuel, then $\beta(t)$ could be such that the trajectory is off of any of the extremals mentioned above, as long as thrust could be made sufficiently large near the end for the constraint to be hit at t_f.

This, however, is only of theoretical interest, since it yields no insight into the structure of general optimal trajectories; also, from a real-world perspective, if σ_v, for example, is larger than anticipated, then following a strategy of waiting for as long as possible to take out the errors (when it is estimated that extra fuel and thrust are available) could lead to mission failure. All indications are that following the constraint the whole way is the minimum fuel solution, which would leave extra fuel for handling unexpected errors near the end of midcourse.

Discrete Burn Solutions

Parameter Optimization Formulation

For the n discrete burn problem, the minimization problem corresponding to Eq. (9) is

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$$\min_{\substack{\Delta v_i,\ i=1,\ldots n \\ \Delta t_i,\ i=1,\ldots n}} E$$

where

$$E = \sigma_v \tau - \sum_{m=1}^{n} \Delta v_m \left(\tau - \sum_{l=1}^{m} \Delta t_l \right),$$

$$j = 1,\ldots,n$$

subject to

$$g_m = \sigma_v \sum_{i=1}^{m} \Delta t_i - \sum_{i=1}^{m} \Delta v_i \left(\tau - \sum_{l=1}^{i} \Delta t_l \right) \geq 0 ,$$

$$m = 1,\ldots,n$$

$$g_{n+1} = \Delta v_T - \sum_{i=1}^{n} \Delta v_i \geq 0 ,$$

$$g_{n+2} = t_f - \sum_{i=1}^{n} \Delta t_i \geq 0 ,$$

and

$$g_{n+2+m} = \Delta v_m \geq 0 ,$$

$$m = 1,\ldots,n$$

where Δv_T is the total amount of available velocity correction corresponding to burning until the divert fuel reaches $m_{\min}$; so

$$\Delta v_T = u_e \cdot ln\left(\frac{m_0}{m_{\min}} \right) .$$

Hereafter, it will be tacitly assumed that the constraints g_{n+2+m} , $m = 1,\ldots,n$, are satisfied.

Taking partial derivatives to apply the usual necessary conditions for parameter optimization,

$$\frac{\partial E}{\partial \Delta v_k} = -\tau + \sum_{l=1}^{k} \Delta t_l ,$$

$$\frac{\partial E}{\partial \Delta t_k} = \sum_{m=k}^{n} \Delta v_m ,$$

$$\frac{\partial g_m}{\partial \Delta v_k} = \begin{cases} -\tau + \sum_{l=1}^{k} \Delta t_l & ,\ k \leq m \\ 0 & ,\ k > m \end{cases}$$

$$\frac{\partial g_m}{\partial \Delta t_k} = \begin{cases} \sigma_v + \sum_{l=1}^{m} \Delta v_l & ,\ k \leq m \\ 0 & ,\ k > m \end{cases}$$

$$\frac{\partial g_{n+1}}{\partial \Delta v_k} = -1 ,$$

$$\frac{\partial g_{n+1}}{\partial \Delta t_k} = 0 ,$$

$$\frac{\partial g_{n+2}}{\partial \Delta v_k} = 0 ,$$

and

$$\frac{\partial g_{n+2}}{\partial \Delta t_k} = -1 .$$

The necessary conditions are

$$\frac{\partial E}{\partial \Delta v_k} + \sum_{m=1}^{n+2} \lambda_m \frac{\partial g_m}{\partial \Delta v_k} = 0$$

and

$$\frac{\partial E}{\partial \Delta t_k} + \sum_{m=1}^{n+2} \lambda_m \frac{\partial g_m}{\partial \Delta t_k} = 0 .$$

Upon substitution of the partials above, these become

$$-\tau + \sum_{l=1}^{k} \Delta t_l + \sum_{m=k}^{n} \lambda_m \left(-\tau + \sum_{l=1}^{k} \Delta t_l \right)$$

$$-\lambda_{n+1} = 0 \qquad (38)$$

and

$$\sum_{l=k}^{n} \Delta v_l + \sum_{m=k}^{n} \lambda_m \left(\sigma_v + \sum_{l=k}^{m} \Delta v_l \right)$$

$$-\lambda_{n+2} = 0 . \qquad (39)$$

Special Cases: n=1 and n=2

For $n = 1$ and $n = 2$, the necessary conditions Eq.s (38) and (39) can be solved directly analytically. For $n = 1$, if

$$\Delta v_T \geq \sigma_v \frac{t_f}{\Delta t_h}$$

(that is, if there is more than sufficient fuel), then apply

$$\Delta v_1 = \sigma_v \frac{t_f}{\Delta t_h}$$

at

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$$\Delta t_1 = t_f \ .$$

Otherwise, apply

$$\Delta v_1 = \Delta v_T$$

at

$$\Delta t_1 = \frac{\Delta v_T}{\sigma_v + \Delta v_T} \tau \ .$$

For $n = 2$, the fuel-limited case has a unique solution

$$\Delta v_1 = \Delta v_2 = \frac{\Delta v_T}{2}$$

with

$$\Delta t_1 = \frac{\Delta v_T}{2\sigma_v + \Delta v_T} \cdot \tau$$

and

$$\Delta t_2 = \frac{2\,\Delta v_T\,\sigma_v}{(2\sigma_v + \Delta v_T)^2} \cdot \tau \ .$$

That is, the solution is to equally divide up all the velocity corrections, and wait to do each burn until the amount of correction estimated by the filter just equals the amount of divert for each burn.

General Fuel-Limited Solution

It turns out that if a guess for the Δv_i is given, then the following computations can be made. First, λ_k can be computed by solving Eq. (39) for it, yielding

$$\lambda_k = \frac{-\left\{ \sum_{l=k}^{n}\Delta v_l + \sum_{m=k+1}^{n} \lambda_m \left(\sigma_v + \sum_{l=k}^{m}\Delta v_l \right) \right\}}{\sigma_v + \Delta v_k}$$

$$k = n, n-1, \ldots, 1 \ . \qquad (40)$$

Δt_k can be obtained by solving $g_m = 0$, $m = 1, \ldots, n$, for it:

$$\Delta t_k = \left\{ -\sigma_v \sum_{l=1}^{k-1}\Delta t_l + \Delta v_k \left(\tau - \sum_{l=1}^{k-1}\Delta t_l \right) \right.$$

$$\left. + \sum_{i=1}^{k-1}\Delta v_i \left(\tau - \sum_{l=1}^{i}\Delta t_l \right) \right\} \Big/ (\sigma_v + \Delta v_k) \quad (41)$$

$$k = 1, 2, \ldots, n \ .$$

Then, with $k = n$ in Eq. (38),

$$\lambda_{n+1} = \left(-\tau + \sum_{l=1}^{n}\Delta t_l \right)\left(1 + \lambda_n \right) \ .$$

Then using this numerical value in Eq. (38) for the rest of the indices, compute

$$q_k = \left(-\tau + \sum_{l=1}^{k}\Delta t_l \right)\left(1 + \sum_{l=k}^{n}\lambda_l \right) - \lambda_{n+1}$$

$$k = n-1, n-2, \ldots, 1 \qquad (42)$$

and check that

$$q_k = 0, \qquad k = n-1, \ldots, 1 \ . \qquad (43)$$

Also, it must be checked that

$$\lambda_k \le 0, \qquad k = 1, 2, \ldots, n+1 \ . \qquad (44)$$

Taking the cue from the n=2 solution, a reasonable guess is that

$$\Delta v_i = \frac{\Delta v_T}{n} \ , \qquad i = 1, \ldots, n \ .$$

This does indeed satisfy Eq.s (43) and (44). The corresponding Δt_i are obtained from Eq. (41).

So, as in the two burn case, the optimal solution is to equally divide up all the velocity corrections, and wait to do each burn until the amount of correction estimated by the filter just equals the amount of divert for each burn.

More-than-Sufficient Fuel Results

If the interceptor has more than sufficient fuel, then $g_{n+2} = 0$ and $g_{n+1} > 0$, so that $\lambda_{n+2} < 0$ and $\lambda_{n+1} = 0$. Eq. (38) then becomes for $k = n$

$$(-\tau + t_f)(\lambda_n + 1) = 0 \ \Rightarrow \ \lambda_n = -1 \ .$$

With $k = n - 1$, Eq. (38) is

$$(-\tau + t_f - \Delta t_n)(\lambda_n + \lambda_{n-1} + 1) = 0$$

$$\Rightarrow \lambda_{n-1} = 0 \ .$$

Successively substituting for $n - 2, \ldots, 1$ yields that

$$\lambda_{n-1} = \lambda_{n-2} = \ldots = \lambda_1 = 0 \ .$$

Next, substituting these λ_i into Eq. (39), beginning with $k = n$,

$$\Delta v_n + \lambda_n(\sigma_v + \Delta v_n) = \lambda_{n+2}$$

$$\Rightarrow \lambda_{n+2} = -\sigma_v \ .$$

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Then, by direct inspection with this value of λ_{n+2} and $\lambda_{n+1} = -1$ used in Eq. (39) with $k < n$, it is seen that the equations are satisfied for all Δv_i.

This is analogous to singular control in the continuous control case. Any history of corrective burns that bring the trajectory onto the constraint at the final time yields an optimal trajectory, since this is the global optimum.

On the other hand, it can be shown that of all the possible solutions for this singular case, the one which yields the minimum fuel expended is to ride the constraint the whole way. That is, if there is more than sufficient fuel, then another optimization problem can be set up, with the same path constraint, plus the additional constraint that the path constraint must be a strict equality constraint at time t_f. Then the cost to be minimized is $-m(t_f)$ (or $\sum \Delta v_i$). The solution to this problem is to keep correcting back to the constraint for all burns. The minimum fuel-expended solution is significant from a practical point of view, because it leaves room for further corrections if, for example, unexpectedly large filter errors are encountered.

Implications of Results

To complete the picture presented above, Fig. 1 depicts the results from the discrete burn section for a variety of numbers of burns, for the same numbers used in examples above: $\sigma_v = 150$ m/s; $t_f = 100\,s$; $\Delta t_h = 10\,s$; $u_e = 2455\,m/s$; $m_0 = 250$ kg; and, $m_{min} = 225$ kg. For comparison, the continuous burn results yield a value of $2942\,m$. Clearly, the discrete burn results are approaching the continuous result as the number of burns approaches infinity.
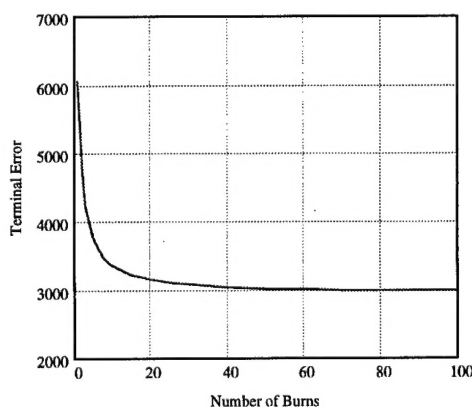

Figure 4. Terminal Error versus Number of Burns

All of the results above, except for the thrust limited but final-mass free case, point to the same basic solution structure. This structure is to fully take out the errors, either continuously, or else as soon as the correction required equals the amount of divert available for the next burn.

Even in the thrust-limited but mass-free case, following the constraint until the thrust level reaches the maximum, then following that maximum, is an extremal. It also typically has only about 1% increase in cost over the bang-bang results. Also, once the mass limit of the interceptor is reached, the bang-bang solution rapidly becomes far less optimal. Also, on a related note, even when the mass limit is not reached, the bang-bang solution burns significantly more fuel than the constraint-following solution.

This means that using the constraint-following extremal is a much more conservative approach, being better equipped to absorb unexpected errors from various sources. If the bang-bang solution happened to have a large performance improvement over the other extremal, it would be significant to incorporate it. As it is, the marginal improvement it provides, combined with the higher margin-of-error risk it entails, probably means that it is only of theoretical interest; the practical result appears to be to use the constraint-following extremal.

As final confirmation of this, it was attempted to find a numerical solution to the discrete burn problem with a large number of burns for the case that yielded a bang-bang solution, but the answer always went to the constraint-following solution. This apparently means that this bang-bang solution disappears when discrete updates, instead of continuous updates, are passed from the remote sensor to the interceptor. In conclusion, even for the one exceptional case, it appears that considering the problem under a variety of performance objectives points to following the constraint until either the maximum thrust level is reached, or else the fuel is exhausted, is the preferred strategy.

Returning to the general implications of the results, from Fig. 4 it is evident that making numerous corrections along the interceptor path, as opposed to making only one or two corrections, significantly improves performance. For the case presented, about 100% fuel increase would be required for the one burn interceptor to

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achieve the same error removal level as 25 to 50 burns (which correspond to a correction every 2 to 4 seconds). This kind of performance juxtaposition is typical for a wide range of values of σ_v, t_f, Δt_h, etc.

Conclusions

It has been shown that an optimal strategy as to when to make midcourse corrections is to continuously, or least frequently, take out the errors in the interceptor path, as the filter associated with the remote sensor makes improvements in the estimate of the threat state propagated to the intercept point. Also, the performance increase in having numerous midcourse corrections, as opposed to only one or two corrections, is large. For a fixed interceptor mass allocation, this means that the probability of successful intercept is much higher if numerous corrections are made. For affecting the design, this means that much less fuel would be required if the capability for numerous corrections was built into the design, so that the interceptor payload could in turn be lighter, yielding higher burnout velocity performance for the booster.

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